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Fixed point and ordered metric spaces a review

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Abstract

weare going o discuss the development in fixed point theory in the field of ordered metric space *in this review paper*

Keywords. Fixed point , ordered metric spaces, partial ordered set, weakly increasing mapping.

Introduction

Upto 1890 it was assumed that the evolution of fixed point theory is in the method of successive approximations whichwas helpful in finding the solutions of differential equations by Charles Emile Picard. But by the beginning of twentieth century it was considered as an important part of analysis. However, from historical point of view, it is assumed that the major classical result in fixed point theory was given by d to L. E. J. Brouwer in 1912.However the publication of Banach Principle by great Polish Mathematician Stefan Banach gives an effective way to find the fixed point of a map.It gives realities and uniqueness for a self mapping in a metric space in which every cauchy sequence is converges to some point in the given matrix. This all has been comprehensively contemplated and summed up in numerous ways from [1 - 4].

It was Caristi who in 1976 introduced the concept of ordered metric space. He in his paper defined an ordered relation and proved a theorem by using a functional satisfying some specific conditions in a metric space.

Presence of fixed points has been considered as of late in [5], and a few speculations of the aftereffect of [5] are given in [6– 10]. Likewise, in [5] a few applications to framework conditions are introduced in [7, 8].A few applications to intermittent limit esteem issue and to some specific issues areseparately given. In 2008, O'Regan and Petrusel [10] gave some existence results for solving Fredholm and Volterraintegral equations. In a portion of the above works, the fixed point results are given for non decreasingfunctions. In 2010, I Altun and H. Simsek [11] presented some fixed point results in ordered metric spaces for non decreasing and weakly increasing functions.They gave an existence theorem for solution of two integral equations.In 2013, PoonamKumam et al [12] introduced some fixed point theorems using non linear contraction conditions which are helpful in solving an integral equation.

Recently A R. Butt et al [13],have given some fixed point results which have been observed to be applicable in homotopy.

Definition and Preliminaries.

We begin this section by some notations, definitions and theorems required in our subsequent discussions.

Definition ([14]) : Let A be a non-empty set. Then $a \in A$ is called a fixed point of a mapping $g: A \rightarrow A$ if $g(a) = a$

Definition ([13]) : Consider a set B such that $B \neq \phi$. Then $(B, d, \preceq)$ is called an ordered (partial) metric space if

(i)$(B, \preceq)$ is a partially ordered set and (ii) (B, d) is a metric space.

Definition:A partially ordered set or poset is a set P and a binaryrelation$\leq$, such that for all $a, b, c \in P$

1 a≤ a (reflexivity).

2 a ≤ b and b ≤ c implies a ≤ c (transitivity).

3 a ≤ b and b ≤a implies a = b. (anti-symmetry).

Definitionsee [15,16]: Let (Z, ⪯) be a partially ordered set. Two functionsP,Q:Z→Z are said to be weakly increasing if Pz⪯ QPz and Qz⪯ PQz for all z∈ Z.

Note that two weakly increasing functions need not be non decreasing

Contribution of various researchers in ordered metric space:

1.IshakAltun and HakanSimsek [11] :

To prove the main results in ordered metric space, authors introduced implicit relation with following three conditions:

S_1 : $S(u_1,u_2,u3,\ldots,u_6)$ is decreasing in variables $u_1,u_2,u3,\ldots,u_6$

S_2 : ∃aright continuous function g defined on set of positive reals where g(0)=0 and g(u) < u for u > 0 such that for v ≥ 0

S(v,w,v,w,0,v+w) ≤ 0

Or

S(v,w,0,0,w,w)≤ 0

This implies v ≤g(w)

S_3 : S(v,0,v,0,0,v) > 0 for every v > 0

The authors proved a lemma in which they proved that$\lim_{n\to\infty} g(u)=0$ when g^n=gogog….og (n-times) and g(u) < u for u > 0

The main result of authors is :

Theorem1.1 : Consider a poset (Y ,≼). Let d is a distance function on Y and (Y,d) is a metric space such that every cauchysequence in Y converges to some point in Y. Let G: Y→Y is an increasing functionsuchthat for every y,z∈ Y with zis partially less or equal to y and

U(d(Gy,Gz),d(y,z),d(y,Gy),d(z,Gz),d(y,Gz),d(z,Gy)) ≤ 0

Where U ∈ζ

then G is continous function.

Or if $\{y_n\}$ proper subset of Y isa increasing sequence with $y_n$approaches to y in Y and

Y_n partially less or equal to y for all nholds if∃y_0 ∈ Y such thaty_0 partially less or equal to $G(y_0)$.Then G has a fixed point.

Corollary1.2 :Consider a poset (Y ,≼). Let d is a distance function on Y and (Y, d) is a metric space such that every cauchy sequence in Y converges to some point in Y . Let G:

Y→Y is an increasing function such that for every y,z∈ Y with z partially less or equal to y and

$D(Gy,Gz) \leq \beta \max\{d(y,z),d(y,Gy),d(z,Gz)\}+(1-\beta)[cd(y,Gz)+ed(z,Gy)$

Where $0\leq\beta\leq1$, $0\leq c\leq½$, $0\leq e\leq1/2$. Then G is continuous function

Or if $\{y_n\}$ proper subset ofY isa increasing sequence with $y_n$approaches to y in Y and

$y_n$partially less or equal to y for all n holds if $\exists y_0 \in Y$ with y_0 partially less or equal to $G(y_0)$.Then G has a fixed point.

Theorem 1.3: Consider a poset (Y, ⪯).Let there is a distance function d on X and (Y,d) is metric space such that every cauchy sequence in Y converges to some point in Y.. Suppose H, I : Y → Y are two functionssuchthatHy is partially less or equal toIHy and Iy is partially less or equal to HIy and for every y,z Ywhere x ≤ y or y ≤ x

$V(d(Hy, Iz), d(y, z), d(y, Hy), d(z, Iz), d(y, Iz), d(z, Iz)) \leq 0$,

where $V \in\zeta$. Then

H is continous function.

or

I is continuous function.

or

if $\{y_n\}$ is proper subset of y is a non decreasing sequence with $y_n$approaching to y in Y

andy_n partially less or equal toy for every n hold

then H and I have a common fixed point

Corollary 1.4: Consider a poset (Y, ⪯) .Consider a complete metric space(X,d) on X. Consider two functions H,I :Y→Y such that for every y,z ∊ X where x ≤ y or y ≤ x

$d(Hy, Iz) \leq g(d(y, z))$,

where g is a function defined on positive realswith right hand limit and functional values of domain as same such that $g(0) = 0$, $g(u) < u$ for tu> 0.

Then

H is continuous function

or I is continuous function

or

if $\{Y_n\}$ proper subset of Y is a increasing sequence with y_n approaching to y in Y,

theny_n partially less or equal toy for all n hold, then H and I have a common fixed point.

2.PoonamKumam et al [12]:

In order to prove main results authors introduced non linear contraction condition specified by a rational expression.

The main results of authors is :

Theorem 2.1. Consider adistance function d on set Y such that every cauchy sequence in Y converges to some point inYand (Y, ⪯) is aposet provided with metric d on it. Consider two functions P, Q : Y → Y satisfying (for pairs (y, z) □ Y × Ysuch that Py ≤ Qz or Qz ≤ Py)the condition :

l (Qy, Qz) ≤γ{l (Py, Qy) ,l(Pz, Qz)/1+l(Py, Pz)} 1+δ (Py,z) ,

whereγandδare positive reals with γ+δ is less than 1.

If the and the functions P ,Q and set Y preserve following two points :

(i)ForY∃ an increasing sequence $\{u_n\}$w.r.t relation⪯, in such a manner that $u_n$approaches u as n tend to∞ and u belongs to Y.This will imply that u_n is partially less or equal to u for all n belonging to set of natural numbers. .Functions P andQaresuch that for every y belonging to Y, Py is partially less or equal to Pz for all z belonging to Q^{-1}(Py)

(ii) the pair (P, Q) of self mappingsis such that P*Q= Q*P and if ∃$\{u_n\}$

in Y such that QPu_napproaches Qu as ntends to ∞ and agrees with

$lt_{n\to\infty}Qu_n = lt_{n\to\infty}Pu_n$= u where u belongs to Y

Then the functionsPand Q will have a point in their common domain with same image i.e., ∃v □ Y such that Pv = Qv.

Corollary 2..2. Consider a poset (Y, ⪯) Letdbea distance function onnon empty set Y such that in ordered pair(Y,d) every Cauchy sequence converges to some point in .SupposeP, Q :Y→ Yare twofunctions satisfying (for pairs (y, z) □ Y×Y such that Py ≤ Qz or Qz ≤ Py) the condition :

l (Qy, Qz) ≤ γl (Py, Pz) ,

whereγ is a positive real number such thatβ is less than 1.

If thefunctions P ,Q and set Y preserve following two points :

(i)For Y ∃ an increasing sequence $\{u_n\}$w.r.t relation ⪯, in such a manner that $u_n$approaches u as n tends to ∞ and u belongs to Y. This will imply that u_n is partially less or equal to u for all n belonging to set of natural numbers. Functions P and Q are such that for every y belonging to Y, Py is partially less or equal to Pz for all z belonging to Q^{-1}(Py)

(ii) The pair (P, Q) of self mappings is such that P*Q= Q*P and if ∃$\{u_n\}$

in Y such that QPu_n approaches Qu as n tends to ∞ and agrees with

$lt_{n\to\infty}Qu_n = lt_{n\to\infty}Pu_n$= u where u belongs to Y

Then the functions P and Q will have a point in their common domain with same image i.e., ∃v □ Y such that Pv = Qv.

Theorem 2.3. Let d be aposet(Y,⪯).Let Y be any non empty setsuchthatinthe ordered pair(Y, d) every cauchy sequence converges to some point in Y. Consider two self functions P: X → X and Q:X → X satisfying (pairs (y,z) □ Y× YsuchthatPy≤Qz or Qz ≤ Py)the condition :

(Qy, Qz) ≤ γ{ (Py, Qy).l(Pz, Qz)/1+(Py,Pz)}+ δ (Rx, Ry)

whereγ, δarepositive realsandγ+δis less than 1.

If thefunctionsP ,Q and set Y preserve following two points :

(i)Functions P and Q are such that for every y belonging to Y, Py is partially less or equal to Pz for all z belonging to Q^{-1}(Py)

(ii)For the pair (P, Q) of self mappings∃{u_n}in Y such that distance between QPu_nandPQu_napproaches zero as n approaches ∞.AlsoQPu_napproaches to Qu as n tends to ∞ and agrees with

$lt_{n\to\infty}Qu_n = lt_{n\to\infty}Pu_n$= u where u belongs to Y

Then the mfunctiond P and Q will have a point in their common domain with same image i.e., ∃v □ Y such that Pv = Qv.

Then thefunctions Q and P have a coincidence point. i.e.,∃w □ Y such that Pw = Qw.

Theorem 2.4. Consider a poset (Y, ⪯) with a distance function$d$ on Y.Let in ordered pair (Y, d) everyCauchy sequence converges to some point in Y where Y is some non empty set. Consider two self functionsP : Y → Y and Q : Y → Y satisfying (for pairs (y, z) □ Y × Y such that Py ≤ Qz or Qz ≤ Py) the condition :

(Qy, TQz) ≤γ{ (y,Qz) · (z, Qz)/1 + l (y, z) }+ δ (x, y)

whereγ, δ are positive reals such that γ+δis less than 1.

If the functionss P ,Q and set Y preserve following two points :

(i)Qyis partially less or equal to Q(Qy)∀y □ Y,

(ii)FunctionsQ should be such that:

(a)it should be defined on each point of its domain

(b) its limit should exist

(c)functional value should be equal to limiting value

(iii) For Y ∃ an increasing sequence {u_n}w.r.t relation ⪯, in such a manner that $u_n$approaches u as n tends ∞ and u belongs to Y. This will imply that u_n is partially less or equal to u for all n belonging to set of natural numbers is regular.

Then, ∃ u in Y suchtat Qu=u.

Theorem 2.5. Consider a poset (Y, ⪯)withadistancefunction$d$ on non empty set Y. Let in ordered pair(Y, d) every Cauchy sequence converges to some point in Y. Consider two self functions P : Y → Y and Q : Y → Y satisfying (for pairs (y, z) □ Y × Ysuch that Py ≤ Qz or Qz ≤ Py) the condition:

l(Qy, Qz) ≤ γ{l (Py, Qy) ·(Pz, Qz)/1+l(Py, Pz)} +δl (Py, Pz)

whereγ, δarepositive reals and γ+δis less than 1.

If the functions P ,Q and set Y preserve following three points :

(i)For Y ∃ an increasing sequence $\{u_n\}$w.r.t relation ⪯, in such a manner that $u_n$approaches u as n tends to ∞ and u belongs to Y. This will imply that u_n is partially less or equal to u for all n belonging to set of natural numbers. Functions P and Q are such that for every y belonging to Y, Py is partially less or equal to Pz for all z belonging to $Q^{-1}(Py)$

(ii) the pair (P, Q) of self mappings is such that P*Q= Q*P and if ∃$\{u_n\}$

in Y such that QPu_n approaches Qu as n tends to ∞ and agrees with

$lt_{n\to\infty}Qu_n= lt_{n\to\infty}Pu_n= u$ where u belongs to Y

(iii) Pissuch that for a increasing sequence $P(u_n)$ inY if $P(u_n)$ approaches u as n tends to ∞ then $P(u_n)$ is less or equal to P(u) or P(u) is less or equal to $P(u_n)$ ∀ n n belonging to set of naturals.

Then the functions P and Q have a common fixed point.

3. Uniqueness results in [12]

In this section we examine therequirement under which Theorem 2.1 assuresthe uniqueness of common fixed point.

Theorem 3.1. Besides the assumptions of Theorem 2.1,let every (y, y∗) □Y × Y, ∃bw□ Y such that Qwis upper bound of Qy*and*Qy∗, then for the functionsP and Q∃ u and v in Ysuchthat Pu= u and Qv = v implies u=Pu=Qv= v..

Theorem 3.2. :Besides the assumptionsof Theorem 2.4., suppose for pairs(y, y∗) ∈Y×Y∃ w∈ Y such that Qw is upper bound of Qy*and*Qy*,then for the functions P and Q ∃ u and v in Y such that Pu= u and Qv = v implies u=Pu=Qv= v..

Corollary 3.3. : Besides the assumptions of Theorem 2.1. (or Theorem 2.2.),if that for every pair (y, y∗)□Y×Y there exists bw□Y such that Qw is upper bound of Qy and Qy∗.Then∃a point y belonging to Y such that y =Qy and this y is unique.

4.A.R. Butt et al [13]:

To prove the main results on ordered metric space introduced implicit relation withthre conditions which are

H_1 :$H(a_1,\ldots,a_6)$ is decreasing in variables a_5,a_6

H :∃k□ [0,1) such that

$$G(a,b,b,a,a+b, 0) \le 0$$

Or

$$G(a,b,a,b, 0,a+b)\le 0$$

This implies

$$a\le kb$$

H_3 :H(a,0, 0,a,a, 0) > 0, for everya> 0..

The main results of the authors is :

To prove the main results he authors have considered a metric space (A,d) such that every cauchy sequence in A converges to some point in A, where set A is having partial order andordered pair (A, ⪯) is a partially ordered set.

The considerationof contractive condition by authors is as:

H(l(mu, mv),l(u,v),d(u,mu),l(v,mv),l(u,mv),l(v,mu)) ≤ 0. (B)

Theorem 4.1.: Consider a self mappingm on set A be such that ∃ a number k which belongs to semi left closed right open interval 0 and 1 with the contractive condition (1-k)l(u,mu)≤l(u,v) which gives the condition (B) for all elements u,v of A such that eitheru≤ v or v≤u and for some H□a.If the set A and function m preserve the following points:

i)∃u_0∈A such that u_0 ≤ m

ii)ifu,v∈A is such that u≤ v then mu≤ m

iii)ifu_n is any sequence in A and u_n approaches u and for all terms of{u_n} such thateitheru ≤ v or v ≤ u where u and v are any two terms of {u_n }, then u_n is partially less or equal to u∀ n.

Then ∃ a point u in set A such that u if fixed point of m.

Theorem 4.2.: Consider a self mapping m on set A be such that ∃ a number k which belongs to semi left closed right open interval 0 and 1 with the contractive condition (1-k)l(u,mu) ≤ l(u,v) which gives the condition (B) for all elements u,v of A such that either u ≤ v or v ≤ u and for some H □ a. If the set A and function m preserve the following points:

i)∃ u_0∈A such that mu_0 ≤ u_0

ii)ifu,v∈A is such that u≤ v then mv≤ m

iii)if u_n is any sequence in A andu_n approaches u and for all terms of{u_n} such that either u ≤ v or v ≤ u where u and v are any two terms of {u_n }, then u_n is partially less or equal to u ∀ n.

Then ∃ a point u inset A such that u if fixed point of m

Theorem 4.3.:Consider a self mapping m on set A be such that ∃ a number k which belongs to semi left closed right open interval 0 and 1 with the contractive condition (1-k)l(u,mu) ≤ l(u,v) which gives the condition (B) for all elements u,v of A such that either u ≤ v or v ≤ u and for some H □ a.
Let function m is either increasing or decreasing. If the set A and function m preserve the following points:

i)∃u_0∈such that mu_0 ≤u_0

ii)if u_n is any sequence in A andu_n approaches u and for all terms of{u_n} such that
either u ≤ v or v ≤ u where u and v are any two terms of {u_n }, then u_n is partially less or equal to u ∀ n

Then ∃ a point u inset A such that u if fixed point of m

Conclusion :

In this paper some fixed point results are given which are quite simple with straight forward proof. The distinctive feature of this paper is that various developments infixed point theory in the field of ordered metric space has been presented step by step. The significance of this Paper is :

i)Existence theorem for common solution of two integral equations has been presented.

ii)Fixed Point results using non linear contraction condition and existence theorem for solution of an integral equation has been presented.

iii)Fixedpoint theorems for uniqueness of fixed point using non linear rational contraction has been presented.

iv)Fixed point results usingspecifiedimplicit relation has been presented which are applicable to homotopy.

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