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L(3,1) Labeling of Some Graph Families of Line Graph of Crown Graph

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Abstract

An $L(3, 1)$ - labeling of a graph G is a function f from the vertex set $V(G)$ to the set of all positive integers such that $|f(x) - f(y)| \geq 3$ if $d(x, y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d(x, y) = 2$ where for all $x, y \in V(G)$. The $L(3, 1)$ labeling number of graph G , denoted by $\lambda(G)$ or λ , is the smallest positive integer m such that $\max\{f(x) : x \in V(G)\} = m$. This paper totally involves the calculation of λ of line graph of crown graphs, duplication of all the vertices of degree two of line graph of a crown graph by an edge, duplication of all the vertices of degree two of line graph of crown graph by a vertex and a graph obtained by connecting two copies of line graph of crown graph by a path.

Keywords: Crown graph, duplication, line graph, $L(3, 1)$ labeling, $\lambda(G)$.

AMSSubjectClassification(2020): 05C78, 05C76.

1. Introduction

An $L(3, 1)$ labeling of graphs is basically motivated from the channel assignment problem which was introduced by Hale [6]. It is also used in solving the frequency assignment problem in most of the wireless network stations. Different frequencies are assigned in such a way so that the transmitter does not interfere with one another. This frequency assignment problem is similar to labeling an $L(3, 1)$ label to each vertex. Throughout we use a simple, finite, undirected graph G , with vertex set $V(G)$ and edge set $E(G)$. For different graph labeling techniques, we use a dynamic survey of graph labeling by Gallian [1]. For various notation and terminology we follow Gross and Yellen [5].

Definition 1.1. For graph G with $w, v \in V(G)$, and for fixed positive integers j and k where $k \leq j$, the function $L : V(G) \rightarrow \mathbb{Z}^+$ is called $L(j, k)$ -labeling of G if and only if $|L(v) - L(w)| \geq j$ if v and w are adjacent and $|L(v) - L(w)| \geq k$ if v and w are at distance two apart. It was first introduced by Griggs and Yeh [4].

Definition 1.2. Let G be a graph with set of vertices $V(G)$ and set of edges $E(G)$. Let f be a function $f : V \rightarrow \mathbb{Z}^+$, where f is $L(3, 1)$ -labeling [3] of G if, for all $u, v \in V(G)$, $|f(u) - f(v)| \geq 3$ if $d(u, v) = 1$ and $|f(u) - f(v)| \geq 1$ if $d(u, v) = 2$.

Definition 1.3. The line graph $L(G)$ [7] of a graph G has vertices expressive edges of G , and two vertices are nearby in $L(G)$ if and only if analogous edges are nearby in G .

Definition 1.4. The crown graph Cr_n [1] is obtained by joining pendant vertices $\{v_i, i = 0, 1, 2, \dots, k-1\}$ with each consecutive vertex $\{u_i, i = 0, 1, 2, \dots, n-1\}$ of the cycle graph C_n . The edge set of crown graph is $\{u_i u_{i+1} = u_i, i = 0, 1, 2, \dots, k-2\} \cup \{u_{n-1} u_n = u_{n-1}\} \cup \{u_i v_i = v_i, i = 0, 1, 2, \dots, k-1\}$.

Definition 1.5. The line graph of crown graph $L(Cr_n)$ is constructed from Cr_n . A cycle $u_0, u_1, u_2, \dots, u_{n-1}, u_0$ is formed using the edges of Cr_n . Each vertex v_i is adjacent to u_i and u_{i+1} , where i is considered modulo $k-1$. Thus, $|V(L(Cr_n))| = 2k$ and $|E(L(Cr_n))| = 3k$. Throughout the article, we use same labels for further investigations.

Definition 1.6. Duplication of u vertices by a new edge $e = u'u''$ in a graph G generates a new graph G' such that $N_G(u') = \{u, u''\}$ and $N_{G'}(u'') = \{u, u'\}$.

Definition 1.7. Duplication of vertex u by a new vertex v forms a new graph G' such that $N_G(u) = N_{G'}(v)$, where $N_G(u)$ is the set of all the vertices adjacent to u in graph G .

2. Results

Theorem 2.1. Line graph of crown graph is an $L(3,1)$ labeled graph and $\chi(L(Cr_n)) = 10$ for $k \geq 3$.

Proof. Define a function $f: V(G) \rightarrow \mathbb{Z}^+$, where $G = L(Cr_n)$, let the consecutive edges and vertices of crown graph Cr_n be e_i and u_i where $0 \leq i \leq k-1$ respectively of C_n . Let u_i and u_{i+1} be joined with a vertex v_i to attain the $L(Cr_n)$. Thus, $V(L(Cr_n)) = \{u_i, v_i; i = 0, 1, 2, \dots, k-1\}$. Thus, following subsequent cases arise on applying $L(3,1)$ labeling on $L(Cr_n)$:

Case 1: $n \equiv 0 \pmod{4}$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, & i = 4k+1, & k=0,1,2,\dots, & \frac{k-4}{4}; \\ 3, & \text{if } x = u_i, & i = 4k+2, & k=0,1,2,\dots, & \frac{k-4}{4}; \\ 6, & \text{if } x = u_i, & i = 4k+3, & k=0,1,2,\dots, & \frac{k-4}{4}; \\ 9, & \text{if } x = u_i, & i = 4k, & k=1,2,\dots, & \frac{k}{4}; \\ 7, & \text{if } x = v_i, & i = 4k+1, & k=0,1,2,\dots, & \frac{k-4}{4}; \\ 10, & \text{if } x = v_i, & i = 4k+2, & k=0,1,2,\dots, & \frac{k-4}{4}; \\ 1, & \text{if } x = v_i, & i = 4k+3, & k=0,1,2,\dots, & \frac{k-4}{4}; \\ 5, & \text{if } x = v_i, & i = 4k, & k=1,2,\dots, & \frac{k}{4} \end{cases}$$

Case 2: $n \equiv 1 \pmod{4}$.

Subcase 2.1: $n=5$.

$$f(x) = \begin{cases} 3(i-1), & \text{if } x = u_i, & i=1,2,3,4; \\ 4, & \text{if } x = u_5; \\ 8, & \text{if } x = v_1; \\ 10, & \text{if } x = v_2; \\ 2, & \text{if } x = v_3; \\ 1, & \text{if } x = v_4; \\ 7, & \text{if } x = v_5. \end{cases}$$

Subcase 2.2: $k > 5$

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, & i = 4k+1, & k=0,1,2,\dots, & \frac{k-5}{4}; \\ 3, & \text{if } x = u_i, & i = 4k+2, & k=0,1,2,\dots, & \frac{k-5}{4}; \\ 6, & \text{if } x = u_i, & i = 4k+3, & k=0,1,2,\dots, & \frac{k-5}{4}; \\ 9, & \text{if } x = u_i, & i = 4k, & k=1,2,\dots, & \frac{k}{4}; \\ 4, & \text{if } x = u_i, & i = 4k+1, & k=0,1,2,\dots, & \frac{k-5}{4}; \\ 8, & \text{if } x = v_i, & i = 4k+2, & k=0,1,2,\dots, & \frac{k-5}{4}; \\ 10, & \text{if } x = v_i, & i = 4k+3, & k=0,1,2,\dots, & \frac{k-5}{4}; \\ 1, & \text{if } x = v_i, & i = 4k, & k=1,2,\dots, & \frac{k}{4}; \\ 5, & \text{if } x = v_i, & i = 4k+1, & k=0,1,2,\dots, & \frac{k-5}{4}; \\ 7, & \text{if } x = v_n. \end{cases}$$

Case3: $n \equiv 2 \pmod{4}$.

Subcase3.1: $n \equiv 6 \pmod{12}$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, & i = 3k+1, & k = 0, 1, 2, \dots, & \frac{k-3}{3}; \\ 3, & \text{if } x = u_i, & i = 3k+2, & k = 0, 1, 2, \dots, & \frac{k-3}{3}; \\ 6, & \text{if } x = u_i, & i = 3k, & k = 1, 2, \dots, & \frac{k}{3}; \\ 7, & \text{if } x = v_i, & i = 3k+1, & k = 0, 1, 2, \dots, & \frac{k-3}{3}; \\ 9, & \text{if } x = v_i, & i = 3k+2, & k = 0, 1, 2, \dots, & \frac{k-3}{3}; \\ 10, & \text{if } x = v_i, & i = 3k, & k = 1, 2, \dots, & \frac{k}{3}. \end{cases}$$

Subcase3.2: $k \equiv 10 \pmod{12}$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, & i = 3k+1, & k = 0, 1, 2, \dots, & \frac{k-4}{3}; \\ 3, & \text{if } x = u_i, & i = 3k+2, & k = 0, 1, 2, \dots, & \frac{k-4}{3}; \\ 6, & \text{if } x = u_i, & i = 3k, & k = 1, 2, \dots, & \frac{k}{3}; \\ 9, & \text{if } x = u_n; \\ 7, & \text{if } x = v_i, & i = 3k+1, & k = 0, 1, 2, \dots, & \frac{k-4}{3}; \\ 9, & \text{if } x = v_i, & i = 3k+2, & k = 0, 1, 2, \dots, & \frac{k-4}{3}; \\ 10, & \text{if } x = v_i, & i = 3k, & k = 1, 2, \dots, & \frac{k-4}{3}; \\ 10, & \text{if } x = v_{n-2}; \\ 1, & \text{if } x = v_{n-1}; \\ 4, & \text{if } x = v_n. \end{cases}$$

Subcase3.3: $k \equiv 2 \pmod{12}$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, & i = 3k+1, & k = 0, 1, 2, \dots, & \frac{k-5}{3}; \\ 3, & \text{if } x = u_i, & i = 3k+2, & k = 0, 1, 2, \dots, & \frac{k-5}{3}; \\ 6, & \text{if } x = u_i, & i = 3k, & k = 1, 2, \dots, & \frac{k-2}{3}; \\ 9, & \text{if } x = u_{n-1}; \\ 4, & \text{if } x = u_n; \\ 7, & \text{if } x = v_i, & i = 3k+1, & k = 0, 1, 2, \dots, & \frac{k-5}{3}; \\ 9, & \text{if } x = v_i, & i = 3k+2, & k = 0, 1, 2, \dots, & \frac{k-5}{3}; \\ 10, & \text{if } x = v_i, & i = 3k, & k = 1, 2, \dots, & \frac{k-5}{3}; \\ 10, & \text{if } x = v_{n-3}; \\ 2, & \text{if } x = v_{n-2}; \\ 1, & \text{if } x = v_{n-1}; \\ 8, & \text{if } x = v_n. \end{cases}$$

Case4: $n \equiv 3 \pmod{4}$.

Subcase4.1: $k=3$.

$$f(x) = \begin{cases} 3(i-1), & \text{if } x = u, & i = 1, 2, 3; \\ 7, & \text{if } x = v; \\ 9, & \text{if } x = v; \\ 10, & \text{if } x = v; \end{cases}$$

Subcase4.2: $k \equiv 7 \pmod{12}$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, & i = 3k+1, & k = 0, 1, 2, \dots, & \frac{k-4}{3}; \\ 3, & \text{if } x = u_i, & i = 3k+2, & k = 0, 1, 2, \dots, & \frac{k-4}{3}; \\ 6, & \text{if } x = u_i, & i = 3k, & k = 1, 2, \dots, & \frac{k-1}{3}; \\ 9, & \text{if } x = u_n; \\ 7, & \text{if } x = v_i, & i = 3k+1, & k = 0, 1, 2, \dots, & \frac{k-4}{3}; \\ 9, & \text{if } x = v_i, & i = 3k+2, & k = 0, 1, 2, \dots, & \frac{k-7}{3}; \\ 10, & \text{if } x = v_i, & i = 3k, & k = 1, 2, \dots, & \frac{k-4}{3}; \\ 10, & \text{if } x = v_{n-2}; \\ 1, & \text{if } x = v_{n-1}; \\ 4, & \text{if } x = v_n. \end{cases}$$

Subcase4.3: $k \equiv 11 \pmod{12}$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, & i = 3k+1, & k = 0, 1, 2, \dots, & \frac{k-5}{3}; \\ 3, & \text{if } x = u_i, & i = 3k+2, & k = 0, 1, 2, \dots, & \frac{k-5}{3}; \\ 6, & \text{if } x = u_i, & i = 3k, & k = 1, 2, \dots, & \frac{k-2}{3}; \\ 9, & \text{if } x = u_{n-1}; \\ 4, & \text{if } x = u_n; \\ 7, & \text{if } x = v_i, & i = 3k+1, & k = 0, 1, 2, \dots, & \frac{k-5}{3}; \\ 9, & \text{if } x = v_i, & i = 3k+2, & k = 0, 1, 2, \dots, & \frac{k-8}{3}; \\ 10, & \text{if } x = v_i, & i = 3k, & k = 1, 2, \dots, & \frac{k-5}{3}; \\ 10, & \text{if } x = v_{n-3}; \\ 2, & \text{if } x = v_{n-2}; \\ 1, & \text{if } x = v_{n-1}; \\ 8, & \text{if } x = v_n. \end{cases}$$

Subcase4.4: $n \equiv 3 \pmod{12}$ and $n > 3$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, & i = 3k+1, & k = 0, 1, 2, \dots, & \frac{k-3}{3}; \\ 3, & \text{if } x = u_i, & i = 3k+2, & k = 0, 1, 2, \dots, & \frac{k-3}{3}; \\ 6, & \text{if } x = u_i, & i = 3k, & k = 1, 2, \dots, & \frac{k}{3}; \\ 7, & \text{if } x = v_i, & i = 3k+1, & k = 0, 1, 2, \dots, & \frac{k-3}{3}; \\ 9, & \text{if } x = v_i, & i = 3k+2, & k = 0, 1, 2, \dots, & \frac{k-3}{3}; \\ 10, & \text{if } x = v_i, & i = 3k, & k = 1, 2, \dots, & \frac{k}{3}. \end{cases}$$

Thus, by all the above cases it is clear that $L(3, 1)$ labeling is satisfied for $L(Cr_n)$ and $\chi(L(Cr_n))$ for each case is 10.

Illustration 2.2. $L(3, 1)$ labeling of $L(Cr_5)$ is shown in the figure below for $\chi = 10$.

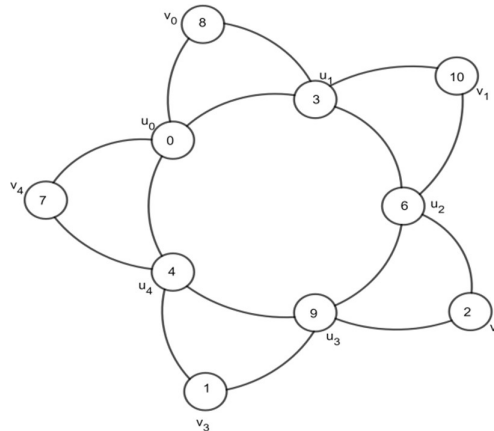


Figure1: $L(3,1)$ labelingof $L(Cr_5)$

Theorem 2. 3. The graph G obtained by duplication of all the vertices of degree two of $L(Cr_n)$ by an edge admits $L(3,1)$ labeling for $k \geq 3$ and $\chi(G) = 10$.

Proof. The graph G obtained by joining all the outer vertices v_i , $i = 0, 1, 2, \dots, n-1$ of $L(Cr_n)$ by an edge having the end vertices w_i and w_{i+1} , where i is considered as modulo $k-1$. Thus $|V(G)| = 4k$ and $|E(G)| = 6k$. Define a function $f: V(G) \rightarrow \mathbb{Z}^+$ such that:

Case 1: $n \equiv 0 \pmod{3}$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, & i = 3k+1, & k = 0, 1, 2, \dots, & \frac{k-3}{3}; \\ 3, & \text{if } x = u_i, & i = 3k+2, & k = 0, 1, 2, \dots, & \frac{k-3}{3}; \\ 6, & \text{if } x = u_i, & i = 3k, & k = 1, 2, \dots, & \frac{k}{3}; \\ 7, & \text{if } x = v_i, & i = 3k+1, & k = 0, 1, 2, \dots, & \frac{k-3}{3}; \\ 9, & \text{if } x = v_i, & i = 3k+2, & k = 0, 1, 2, \dots, & \frac{k-3}{3}; \\ 10, & \text{if } x = v_i, & i = 3k, & k = 1, 2, \dots, & \frac{k}{3}; \\ 1, & \text{if } x = w_i, & i = 2k+1, & k = 0, 1, 2, \dots, k-1; \\ 4, & \text{if } x = w_i, & i = 2k, & k = 1, 2, \dots, k. \end{cases}$$

Case 2: $k \equiv 1 \pmod{3}$.

Subcase 2.1: $k = 4$.

$$f(x) = \begin{cases} 3(i-1), & \text{if } x \in \{u_1, u_2, u_3, u_4\}; \\ 7, & \text{if } x = v_1; \\ 10, & \text{if } x = v_2; \\ 1, & \text{if } x \in \{v_3, w_1, w_3, w_8\}; \\ 4, & \text{if } x \in \{v_4, w_2, w_4, w_6\}; \\ 8, & \text{if } x \in \{w_{2n-3}, w_{2n-1}\}. \end{cases}$$

Subcase 2.2: $k > 4$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, & i = 3k+1, & k = 0, 1, 2, \dots, \frac{k-4}{3}; \\ 3, & \text{if } x = u_i, & i = 3k+2, & k = 0, 1, 2, \dots, \frac{k-4}{3}; \\ 6, & \text{if } x = u_i, & i = 3k, & k = 1, 2, \dots, \frac{k-1}{3}; \\ 9, & \text{if } x = u_n; \\ 7, & \text{if } x = v_i, & i = 3k+1, & k = 0, 1, 2, \dots, \frac{k-4}{3}; \\ 10, & \text{if } x = v_i, & i = 3k+2, & k = 0, 1, 2, \dots, \frac{k-4}{3}; \\ 9, & \text{if } x = v_i, & i = 3k, & k = 1, 2, \dots, \frac{k-4}{3}; \\ 1, & \text{if } x \in \{v_{n-1}, w_{2n-1}\}; \\ 4, & \text{if } x \in \{v_n, w_{2n-3}\}; \\ 1, & \text{if } x = w_i, & i = 2k+1, & k = 0, 1, 2, \dots, k-3; \\ 4, & \text{if } x = w_i, & i = 2k, & k = 1, 2, \dots, k-2; \\ 8, & \text{if } x \in \{w_{2n-2}, w_{2n}\}. \end{cases}$$

Case 3: $n \equiv 2 \pmod{3}$.

Subcase 3.1: $n = 5$.

$$f(x) = \begin{cases} 3(i-1), & \text{if } x \in \{u_1, u_2, u_3, u_4\}; \\ 4, & \text{if } x \in \{u_5, w_2, w_4\}; \\ 7, & \text{if } x = v_1; \\ 10, & \text{if } x = v_2; \\ 2, & \text{if } x = v_3; \\ 1, & \text{if } x \in \{v_4, w_1, w_3, w_{2n}\}; \\ 5, & \text{if } x \in \{w_{2n-5}, w_{2n-3}, w_{2n-1}\}; \\ 8, & \text{if } x = w_i, & i \in \{v_5, w_{2n-4}, w_{2n-2}\}. \end{cases}$$

Subcase 3.2: $n > 5$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, & i = 3k+1, & k = 0, 1, 2, \dots, \frac{k-5}{3}; \\ 3, & \text{if } x = u_i, & i = 3k+2, & k = 0, 1, 2, \dots, \frac{k-5}{3}; \\ 6, & \text{if } x = u_i, & i = 3k, & k = 1, 2, \dots, \frac{k-2}{3}; \\ 9, & \text{if } x = u_{n-1}; \\ 4, & \text{if } x = u_n; \\ 7, & \text{if } x = v_i, & i = 3k+1, & k = 0, 1, 2, \dots, \frac{k-5}{3}; \\ 10, & \text{if } x = v_i, & i = 3k+2, & k = 0, 1, 2, \dots, \frac{k-5}{3}; \\ 9, & \text{if } x = v_i, & i = 3k, & k = 1, 2, \dots, \frac{k-5}{3}; \\ 2, & \text{if } x = v_{n-2}; \\ 1, & \text{if } x \in \{v_{n-1}, w_{2n}\}; \\ 1, & \text{if } x = w_i, & i = 2k+1, & k = 0, 1, 2, \dots, k-4; \\ 4, & \text{if } x = w_i, & i = 2k, & k = 1, 2, \dots, k-3; \\ 5, & \text{if } x \in \{w_{2n-5}, w_{2n-3}, w_{2n-1}\}; \\ 8, & \text{if } x = w_i, & i \in \{v_5, w_{2n-4}, w_{2n-2}\}. \end{cases}$$

Thus, by above cases it is clear that $L(3, 1)$ labeling is satisfied for duplication of outer vertices of $L(Cr_n)$ by an edge and λ number for each case is 10.

Theorem 2.4. The graph G obtained by duplication of all the vertices of degree two of $L(Cr_n)$ by a vertex admits $L(3, 1)$ labeling with $\lambda(G) = 12$ for $k \equiv 0 \pmod{6}$ and $\lambda(G) = 13$ otherwise.

Proof. Let G be a graph obtained by duplicating each of the vertices of degree two $\{v_i, i = 0, 1, 2, \dots, k-1\}$ by a new vertex $\{w_i, i=0, 1, 2, \dots, k-1\}$, where w_i are adjacent to w_{i+1} and u_{i+1} for $i = 0, 1, 2, \dots, k-1$. Thus, $|V(G)|=3k, |E(G)|=5k$. Define a function $f: V(G) \rightarrow \mathbb{Z}^+$ such that:

Case 1: $k \equiv 0 \pmod{3}$.

Subcase 1.1: $k=3$.

$$f(x) = \begin{cases} 3(i-1), & \text{if } x = u_i, & i=1,2,3; \\ 7, & \text{if } x = v_1; \\ 9, & \text{if } x = v_2; \\ 10, & \text{if } x = v_3; \\ 11, & \text{if } x = w_1; \\ 12, & \text{if } x = w_2; \\ 13, & \text{if } x = w_3. \end{cases}$$

Subcase 1.2: $k \equiv 0 \pmod{6}$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, & i=3k+1, & k=0,1,2,\dots, & \frac{k-3}{3}; \\ 3, & \text{if } x = u_i, & i=3k+2, & k=0,1,2,\dots, & \frac{k-3}{3}; \\ 6, & \text{if } x = u_i, & i=3k, & k=1,2,\dots, & \frac{k}{3}; \\ 7, & \text{if } x = v_i, & i=3k+1, & k=0,1,2,\dots, & \frac{k-3}{3}; \\ 9, & \text{if } x = v_i, & i=3k+2, & k=0,1,2,\dots, & \frac{k-3}{3}; \\ 10, & \text{if } x = v_i, & i=3k, & k=1,2,\dots, & \frac{k}{3}; \\ 11, & \text{if } x = w_i, & i=2k+1, & k=0,1,2,\dots, & 2 \frac{k-2}{2}; \\ 12, & \text{if } x = w_i, & i=2k, & k=1,2,\dots, & \frac{k}{2} \end{cases}$$

Subcase 1.3: $k \equiv 3 \pmod{6}$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, & i=3k+1, & k=0,1,2,\dots, & \frac{k-3}{3}; \\ 3, & \text{if } x = u_i, & i=3k+2, & k=0,1,2,\dots, & \frac{k-3}{3}; \\ 6, & \text{if } x = u_i, & i=3k, & k=1,2,\dots, & \frac{k}{3}; \\ 7, & \text{if } x = v_i, & i=3k+1, & k=0,1,2,\dots, & \frac{k-3}{3}; \\ 9, & \text{if } x = v_i, & i=3k+2, & k=0,1,2,\dots, & \frac{k-3}{3}; \\ 10, & \text{if } x = v_i, & i=3k, & k=1,2,\dots, & \frac{k}{3}; \\ 11, & \text{if } x = w_i, & i=2k+1, & k=0,1,2,\dots, & 2 \frac{k-3}{2}; \\ 12, & \text{if } x = w_i, & i=2k, & k=1,2,\dots, & \frac{k-1}{2}; \\ 13, & \text{if } x = w_n. \end{cases}$$

Case 2: $k \equiv 1 \pmod{3}$.

Subcase 2.1: $k=4$.

$$f(x) = \begin{cases} 3(i-1), & \text{if } x \in \{u_1, u_2, u_3, u_4\}; \\ 7, & \text{if } x = v_1; \\ 10, & \text{if } x = v_2; \\ 1, & \text{if } x = v_3; \\ 4, & \text{if } x = v_4; \\ 11, & \text{if } x = w_1; \\ 12, & \text{if } x \in \{w_2, w_3\}; \\ 13, & \text{if } x = w_4. \end{cases}$$

Subcase2.2: $k \equiv 1 \pmod{6}$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, & i = 3k+1, & k = 0, 1, 2, \dots, \frac{k-4}{3}; \\ 3, & \text{if } x = u_i, & i = 3k+2, & k = 0, 1, 2, \dots, \frac{k-4}{3}; \\ 6, & \text{if } x = u_i, & i = 3k, & k = 1, 2, \dots, \frac{k-1}{3}; \\ 9, & \text{if } x = u_n; \\ 7, & \text{if } x = v_i, & i = 3k+1, & k = 0, 1, 2, \dots, \frac{k-4}{3}; \\ 10, & \text{if } x = v_i, & i = 3k+2, & k = 0, 1, 2, \dots, \frac{k-4}{3}; \\ 9, & \text{if } x = v_i, & i = 3k, & k = 1, 2, \dots, \frac{k-4}{3}; \\ 1, & \text{if } x = v_{n-1}; \\ 4, & \text{if } x = v_n; \\ 11, & \text{if } x = w_i, & i = 2k+1, & k = 0, 1, 2, \dots, \frac{k-3}{2}; \\ 12, & \text{if } x = w_i, & i = 2k, & k = 1, 2, \dots, \frac{k-1}{2}; \\ 13, & \text{if } x = w_n. \end{cases}$$

Subcase2.3: $k \equiv 4 \pmod{6}$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, & i = 3k+1, & k = 0, 1, 2, \dots, \frac{k-4}{3}; \\ 3, & \text{if } x = u_i, & i = 3k+2, & k = 0, 1, 2, \dots, \frac{k-4}{3}; \\ 6, & \text{if } x = u_i, & i = 3k, & k = 1, 2, \dots, \frac{k-1}{3}; \\ 9, & \text{if } x = u_n; \\ 7, & \text{if } x = v_i, & i = 3k+1, & k = 0, 1, 2, \dots, \frac{k-4}{3}; \\ 10, & \text{if } x = v_i, & i = 3k+2, & k = 0, 1, 2, \dots, \frac{k-4}{3}; \\ 9, & \text{if } x = v_i, & i = 3k, & k = 1, 2, \dots, \frac{k-4}{3}; \\ 1, & \text{if } x = v_{n-1}; \\ 4, & \text{if } x = v_n; \\ 11, & \text{if } x = w_i, & i = 2k+1, & k = 0, 1, 2, \dots, \frac{k-4}{2}; \\ 12, & \text{if } x = w_i, & i = 2k, & k = 1, 2, \dots, \frac{k-2}{2}; \\ 13, & \text{if } x = w_{n-1}. \end{cases}$$

Case3: $k \equiv 2 \pmod{3}$.

Subcase3.1: $k \equiv 5 \pmod{6}$.

Subsubcase3.1.1: $k = 5$.

$$f(x) = \begin{cases} 3(i-1), & \text{if } x \in \{u_1, u_2, u_3, u_4\}; \\ 4, & \text{if } x = u_n; \\ 7, & \text{if } x = v_1; \\ 10, & \text{if } x = v_2; \\ 2, & \text{if } x = v_3; \\ 1, & \text{if } x = v_4; \\ 8, & \text{if } x = v_5; \\ 11, & \text{if } x = w_1; \\ 12, & \text{if } x \in \{w_2, w_4\}; \\ 13, & \text{if } x \in \{w_3, w_5\}. \end{cases}$$

Subsubcase 3.1.2: $k > 5$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, & i = 3k+1, & k = 0, 1, 2, \dots, & \frac{k-5}{3}; \\ 3, & \text{if } x = u_i, & i = 3k+2, & k = 0, 1, 2, \dots, & \frac{k-5}{3}; \\ 6, & \text{if } x = u_i, & i = 3k, & k = 1, 2, \dots, & \frac{k-2}{3}; \\ 9, & \text{if } x = u_{n-1}; \\ 4, & \text{if } x = u_n; \\ 7, & \text{if } x = v_i, & i = 3k+1, & k = 0, 1, 2, \dots, & \frac{k-5}{3}; \\ 10, & \text{if } x = v_i, & i = 3k+2, & k = 0, 1, 2, \dots, & \frac{k-5}{3}; \\ 9, & \text{if } x = v_i, & i = 3k, & k = 1, 2, \dots, & \frac{k-5}{3}; \\ 2, & \text{if } x = v_{n-2}; \\ 1, & \text{if } x = v_{n-1}; \\ 8, & \text{if } x = v_n; \\ 11, & \text{if } x = w_i, & i = 2k+1, & k = 0, 1, 2, \dots, & \frac{k-3}{2}; \\ 12, & \text{if } x = w_i, & i = 2k, & k = 1, 2, \dots, & \frac{k-1}{2}; \\ 13, & \text{if } x \in \{w_{n-1}, w_n\}. \end{cases}$$

Subcase 3.2: $k \equiv 2 \pmod{6}$

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, & i = 3k+1, & k=0,1,2,\dots, & \frac{k-5}{3}; \\ 3, & \text{if } x = u_i, & i = 3k+2, & k=0,1,2,\dots, & \frac{k-5}{3}; \\ 6, & \text{if } x = u_i, & i = 3k, & k=1,2,\dots, & \frac{k-2}{3}; \\ 9, & \text{if } x = u_{n-1}; \\ 4, & \text{if } x = u_n; \\ 7, & \text{if } x = v_i, & i = 3k+1, & k=0,1,2,\dots, & \frac{k-5}{3}; \\ 10, & \text{if } x = v_i, & i = 3k+2, & k=0,1,2,\dots, & \frac{k-5}{3}; \\ 9, & \text{if } x = v_i, & i = 3k, & k=1,2,\dots, & \frac{k-5}{3}; \\ 2, & \text{if } x = v_{n-2}; \\ 1, & \text{if } x = v_{n-1}; \\ 8, & \text{if } x = v_n; \\ 11, & \text{if } x = w_i, & i = 2k+1, & k=0,1,2,\dots, & \frac{k-4}{2}; \\ 12, & \text{if } x = w_i, & i = 2k, & k=1,2,\dots, & \frac{k-4}{2}; \\ 13, & \text{if } x = w_{n-1}. \end{cases}$$

Thus, by above all the cases it is clear that the graph G obtained by duplication of inner vertices of degree two of $L(Cr_n)$ by a vertex admits $L(3,1)$ labeling and $\lambda(G) = 12$ for $k \equiv 0 \pmod{6}$ and $\lambda(G) = 13$ otherwise.

Theorem 2.5. The graph G' obtained by connecting two copies of $L(Cr_n)$ for all $k \geq 3$, by a path P_m , where $m \geq 4$ is $L(3,1)$ graph and $\lambda = 11$.

Proof. For $L(Cr_n) | V(L(Cr_n)) | = 2k$ and $|E(L(Cr_n))| = 3k$ and for the graph G' obtained by connecting two copies of $L(Cr_n)$ by a path P_m , where $m \geq 4$ and first and last vertices of P_m , are connected to u_0 of each copy of $L(Cr_n)$. Name vertices of P_m as $p_1, p_2, \dots, p_{m-2}$ other than first and last vertices of P_m . Thus, $|V(G')| = 4k + m - 2$ and $|E(G')| = 6k + m - 1$. We refer to the labeling of $L(Cr_n)$ as in theorem 3.1 with the same function $f: V(G) \rightarrow \mathbb{Z}^+$. Now define a new function $f': V(G') \rightarrow \mathbb{Z}^+$ such that:

Case 1: $m \equiv 0 \pmod{3}$.

$$f'(x) = \begin{cases} f(x), & \text{if } x \in V(L(Cr_n)); \\ 5, & \text{if } x = p_{3k+1}, \quad k=0,1,2,\dots, \quad \frac{m-3}{3}; \\ 8, & \text{if } x = p_{3k+2}, \quad k=0,1,2,\dots, \quad \frac{m-6}{3}; \\ 11, & \text{if } x = p_{3k}, \quad k=1,2,\dots, \quad \frac{m-3}{3}. \end{cases}$$

Case 2: $m \equiv 1 \pmod{3}$.

Subcase 2.1: $m = 4$.

$$f'(x) = \begin{cases} f(x), & \text{if } x \in V(L(Cr_n)); \\ 5, & \text{if } x = p_1; \\ 11, & \text{if } x = p_2. \end{cases}$$

Subcase 2.2: $m > 4$.

$$f(x) = \begin{cases} f(x), & \text{if } x \in V(L(Cr_n)); \\ 5, & \text{if } x = p_{3k+1}, \quad k=0,1,2,\dots, \quad \frac{m-4}{3}; \\ 8, & \text{if } x = p_{3k+2}, \quad k=0,1,2,\dots, \quad \frac{m-4}{3}; \\ 11, & \text{if } x = p_{3k}, \quad k=1,2,\dots, \quad \frac{m-4}{3}. \end{cases}$$

Case3: $m \equiv 2 \pmod{3}$.

$$f(x) = \begin{cases} f(x), & \text{if } x \in V(L(Cr_n)); \\ 5, & \text{if } x = p_{3k+1}, \quad k=0,1,2,\dots, \quad \frac{m-5}{3}; \\ 8, & \text{if } x = p_{3k+2}, \quad k=0,1,2,\dots, \quad \frac{m-5}{3}; \\ 11, & \text{if } x = p_{3k}, \quad k=1,2,\dots, \quad \frac{m-2}{3}. \end{cases}$$

Thus, by above all cases it is clear that the graph G' obtained by connecting two copies of $L(Cr_n)$ for all $k \geq 3$, by a path P_m , where $m \geq 4$, admits $L(3,1)$ labeling and $\lambda = 11$.

3. Conclusion

Line graph of crown graph is constructed and $L(3,1)$ had been applied on it. $L(3,1)$ labeling on $L(Cr_n)$ results in $\lambda = 10$. All the vertices of degree two are duplicated by an edge and the obtained graph admits $L(3,1)$ labeling for $\lambda = 10$. Also $\lambda = 12$ for $k \equiv 0 \pmod{6}$ and $\lambda = 13$ otherwise for the graph obtained by duplicating each of its vertices of degree two by a new vertex. New graph is constructed by joining two copies of $L(Cr_n)$ by a path P_m which admits $L(3,1)$ labeling and $\lambda = 11$. $L(3,1)$ labeling can be applied on more graph families which can be constructed using crown graph and different graph operations like corona product, complement of a graph, shadow graph and many more.

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